CHARACTERIZATION OF GENERALIZED JORDAN HIGHER DERIVATIONS ON TRIANGULAR RINGS

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ABSTRACT. Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings and $\mathcal{M}$ be a $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the associated triangular ring. It is shown that every additive generalized Jordan (triple) higher derivation on $\mathcal{U}$ is a generalized higher derivation.

1. INTRODUCTION

Let $\mathcal{A}$ be a ring (or an algebra over a commutative ring) and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Recall that an additive (linear) map δ from $\mathcal{A}$ into $\mathcal{M}$ is called a derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$; a Jordan derivation if $\delta(A^2) = \delta(A)A + A\delta(A)$ for each $A \in \mathcal{A}$; and a Jordan triple derivation if $\delta(ABA) = \delta(A)BA + A\delta(B)A + AB\delta(A)$ for all $A, B \in \mathcal{A}$. More generally, if there is a derivation $\tau : \mathcal{A} \to \mathcal{M}$ such that $\delta(AB) = \delta(A)B + A\tau(B)$ for all $A, B \in \mathcal{A}$, then δ is called a generalized derivation and τ is the relating derivation; if there is a Jordan derivation $\tau : \mathcal{A} \to \mathcal{M}$ such that $\delta(A^2) = \delta(A)A + A\tau(A)$ for all $A \in \mathcal{A}$, then δ is called a generalized Jordan derivation and τ is the relating Jordan derivation; if there is a Jordan triple derivation $\tau : \mathcal{A} \to \mathcal{M}$ such that $\delta(ABA) = \delta(A)BA + A\tau(B)A + AB\tau(A)$ for all $A, B \in \mathcal{A}$, then δ is called a generalized Jordan triple derivation and τ is the relating Jordan triple derivation.

The structures of derivations, Jordan derivations, generalized derivations and generalized Jordan derivations were systematically studied. It is obvious that every generalized derivation is a generalized Jordan derivation. But the converse is in general not true. Zhu in [17] proved that every generalized Jordan derivation from a 2-torsion free semiprime ring with identity into itself is a generalized derivation. Hou and Qi in [8] proved that every additive generalized Jordan derivation of nest algebras on a Banach space is an additive generalized derivation. For other results, see [1, 2, 7, 11] and the references therein.

On the other hand, higher derivations had been studied. We first recall the concepts about higher derivations and generalized higher derivations.

Definition 1.1. ([6]) *Let $D = (\tau_i)_{i \in \mathbb{N}}$ be a family of additive maps of ring $\mathcal{R}$ such that $\tau_0 = \mathrm{id}_{\mathcal{R}}$. D is said to be:*

a higher derivation (HD, for short) if for every $n \in \mathbb{N}$ we have $\tau_n(AB) = \sum_{i+j=n} \tau_i(A)\tau_j(B)$ for all $A, B \in \mathcal{R}$;

a Jordan higher derivation (JHD, for short) if for every $n \in \mathbb{N}$ we have $\tau_n(A^2) = \sum_{i+j=n} \tau_i(A)\tau_j(A)$ for all $A \in \mathcal{R}$;

a Jordan triple higher derivation (JTHD, for short) if for every $n \in \mathbb{N}$ we have $\tau_n(ABA) = \sum_{i+j+k=n} \tau_i(A)\tau_j(B)\tau_k(A)$ for all $A, B \in \mathcal{R}$.

Definition 1.2. ([12]) *Let $G = (\delta_i)_{i \in \mathbb{N}}$ be a family of additive maps of ring $\mathcal{R}$ such that $\delta_0 = \mathrm{id}_{\mathcal{R}}$. G is said to be:*

a generalized higher derivation (GHD, for short) if there exists a higher derivation $D = (\tau_i)_{i \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$ we have $\delta_n(AB) = \sum_{i+j=n} \delta_i(A)\tau_j(B)$ for all $A, B \in \mathcal{R}$;

a generalized Jordan higher derivation (GJHD, for short) if there exists a Jordan higher derivation $D = (\tau_i)_{i \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$ we have $\delta_n(A^2) = \sum_{i+j=n} \delta_i(A)\tau_j(A)$ for all $A \in \mathcal{R}$;

a generalized Jordan triple higher derivation (GJTHD, for short) if there exists a Jordan triple higher derivation $D = (\tau_i)_{i \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$ we have $\delta_n(ABA) = \sum_{i+j+k=n} \delta_i(A)\tau_j(B)\tau_k(A)$ for all $A, B \in \mathcal{R}$.

M. Ferrero and C. Haetinger in [5] proved that every Jordan higher derivation of a 2-torsion-free ring is a Jordan triple higher derivation and every Jordan triple higher derivation in a 2-torsion-free semiprime ring is a higher derivation. Y. S. Jung in [10] proved that every generalized Jordan triple higher derivation on a 2-torsion-free prime ring is a generalized higher derivation. Recently, Hou and Qi [13] proved that every additive Jordan (triple) higher derivation of nest algebras on a Banach space is a higher derivation. Xiao and Wei in [16] proved that every Jordan higher derivation on triangular algebras is a higher derivation.

In the present paper, we will consider generalized Jordan derivations on triangular rings. In fact, we show that every additive generalized Jordan higher derivation on triangular rings is a generalized higher derivation (Theorem 3.1). By using the result, we prove that every generalized Jordan triple higher derivation on triangular rings is also a generalized higher derivation (Theorem 3.2).

Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings (or algebras over a commutative ring $\mathcal{R}$), and $\mathcal{M}$ be a $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module, that is, for any

$a \in \mathcal{A}$ and $b \in \mathcal{B}$, $a\mathcal{M} = \mathcal{M}b = \{0\}$ imply $a = 0$ and $b = 0$. The $\mathcal{R}$-ring ($\mathcal{R}$-algebra)

$$\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \}$$

under the usual matrix operations is called a triangular ring (algebra), and the idempotent element $P = \begin{pmatrix} I_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix}$ is called the standard idempotent of $\mathcal{U}$. Clearly, $I - P = \begin{pmatrix} 0 & 0 \\ 0 & I_\mathcal{B} \end{pmatrix}$. Here I, $I_\mathcal{A}$ and $I_\mathcal{B}$ are units of $\mathcal{U}$, $\mathcal{A}$ and $\mathcal{B}$, respectively. For more details for triangular rings (algebras) and its relating questions, the reader see [3, 14] and the references therein.

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers including 0.

2. PRELIMINARIES

In this section, we give some preliminaries which are needed in Section 3.

Lemma 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings, and $\mathcal{M}$ be a $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular ring. Assume that $G = (\delta_i)_{i \in \mathbb{N}}$ is an additive generalized Jordan higher derivation of $\mathcal{U}$ and $D = (\tau_i)_{i \in \mathbb{N}}$ the relating additive Jordan higher derivation. Then for all $X, Y \in \mathcal{U}$, the following statements hold:*

(1) $\delta_n(XY + YX) = \sum_{i+j=n}(\delta_i(X)\tau_j(Y) + \delta_i(Y)\tau_j(X))$,
(2) $\delta_n(XYX) = \sum_{i+j+k=n} \delta_i(X)\tau_j(Y)\tau_k(X)$.

Proof. (1) On the one hand, we have

$$\begin{aligned} \delta_n((X+Y)^2) =\ & \sum_{i+j=n} \delta_i(X+Y)\tau_j(X+Y) \\ =\ & \sum_{i+j=n}(\delta_i(X)\tau_j(X) + \delta_i(X)\tau_j(Y) \\ & + \delta_i(Y)\tau_j(X) + \delta_i(Y)\tau_j(Y)), \end{aligned}$$

and on the other hand,

$$\begin{aligned} \delta_n((X+Y)^2) =\ & \delta_n(X^2 + XY + YX + Y^2) \\ =\ & \sum_{i+j=n} \delta_i(X)\tau_j(X) + \delta_n(XY + YX) + \sum_{i+j=n} \delta_i(Y)\tau_j(Y). \end{aligned}$$

Comparing the above two equations, we obtain that

$$\delta_n(XY + YX) = \sum_{i+j=n} (\delta_i(X)\tau_j(Y) + \delta_i(Y)\tau_j(X)).$$

(2) Let $S = \delta_n(X(XY+YX)+(XY+YX)X)$. By [16], $D = (\tau_i)_{i\in\mathbb{N}}$ is a higher derivation. Then, using (1) and the fact, on the one hand, we have

$$
\begin{aligned}
S = {} & \textstyle\sum_{i+j=n}(\delta_i(X)\tau_j(XY+YX)+\delta_i(XY+YX)\tau_j(X)) \\
= {} & \textstyle\sum_{i+j=n}\sum_{r+s=j}\delta_i(X)(\tau_r(X)\tau_s(Y)+\tau_r(Y)\tau_s(X)) \\
& +\textstyle\sum_{i+j=n}\sum_{k+l=i}(\delta_k(X)\tau_l(Y)+\delta_k(Y)\tau_l(X))\tau_j(X) \\
= {} & \textstyle\sum_{i+j=n}\sum_{r+s=j}\delta_i(X)\tau_r(X)\tau_s(Y)+2\sum_{i+j+k=n}\delta_i(X)\tau_j(Y)\tau_k(X) \\
& +\textstyle\sum_{i+j=n}\sum_{k+l=i}\delta_k(Y)\tau_l(X)\tau_j(X);
\end{aligned}
$$

on the other hand,

$$
\begin{aligned}
S = {} & \delta_n(X^2Y+2XYX+YX^2) \\
= {} & \textstyle\sum_{i+j=n}(\delta_i(X^2)\tau_j(Y)+\delta_i(Y)\tau_j(X^2))+2\delta_n(XYX) \\
= {} & \textstyle\sum_{i+j=n}\sum_{r+s=i}(\delta_r(X)\tau_s(X)\tau_j(Y) \\
& +\textstyle\sum_{i+j=n}\sum_{k+l=j}\delta_i(Y)\tau_k(X)\tau_l(X)+2\delta_n(XYX).
\end{aligned}
$$

These two equations imply that (2) is true, completing the proof of the lemma. $\qquad\square$

Now, let P be the standard idempotent of $\mathcal{U}$. For the convenience, in the sequel, let $Q = I - P$. Then $\mathcal{U} = P\mathcal{U}P + P\mathcal{U}Q + Q\mathcal{U}Q$.

By [16], every Jordan higher derivation $D = (\tau_i)_{i\in\mathbb{N}}$ on the triangular ring $\mathcal{U}$ is in fact a higher derivation and satisfies that

$$\tau_n(I) = 0 \quad\text{and}\quad \tau_n(P), \tau_n(Q) \in P\mathcal{U}Q \tag{2.1}$$

for all $n \in \mathbb{N}$. By the definition of higher derivations, we have

$$\tau_n(XY) = \sum_{i+j=n}\tau_i(X)\tau_j(Y) \quad\text{for all}\quad X, Y \in \mathcal{U}. \tag{2.2}$$

Thus, for any $X \in \mathcal{U}$, by Eq.(2.1) and noting that $Q\mathcal{U}P = \{0\}$, we get

$$
\begin{aligned}
\tau_n(PXQ) = {} & \textstyle\sum_{i+j=n}\tau_i(P)\tau_j(PXQ) \\
= {} & P\tau_n(PXQ)+\tau_1(P)\tau_{n-1}(PXQ)+\cdots+\tau_n(P)PXQ \in P\mathcal{U}Q;
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
\tau_n(PXP) = {} & \textstyle\sum_{i+j=n}\tau_i(P)\tau_j(PXP) \\
= {} & P\tau_n(PXP)+\tau_1(P)\tau_{n-1}(PXP)+\cdots+\tau_n(P)PXP \in P\mathcal{U}P+P\mathcal{U}Q
\end{aligned}
\tag{2.4}
$$

and

$$
\begin{aligned}
\tau_n(QXQ) = {} & \textstyle\sum_{i+j=n}\tau_i(Q)\tau_j(QXQ) \\
= {} & Q\tau_n(QXQ)+\tau_1(Q)\tau_{n-1}(QXQ)+\cdots+\tau_n(Q)QXQ \in P\mathcal{U}P+Q\mathcal{U}Q.
\end{aligned}
\tag{2.5}
$$

Remark 2.2. By the above analysis, for any Jordan higher derivation $D = (\tau_i)_{i\in\mathbb{N}}$ on $\mathcal{U}$, we have the following properties:

$\mathbf{P_D}$: (i)$\tau_n(I) = 0$; (ii)$\tau_n(P) \in PUQ$; (iii)$\tau_n(Q) \in PUQ$; (iv) $\tau_n(PUQ) \subseteq PUQ$; (v)$\tau_n(PUP) \subseteq PUP + PUQ$ and $\tau_n(QUQ) \subseteq QUQ + PUQ$ for each $n \in \mathbb{N}$.

For any generalized higher derivation $D = (\delta_i)_{i \in \mathbb{N}}$ on a triangular ring $\mathcal{U}$, by the definition, it is clear that δ_1 is a generalized Jordan derivation and τ_1 the relating Jordan derivation. Hence δ_1 is a generalized derivation by [15], that is, $\delta_1(XY) = \delta_1(X)Y + X\tau_1(Y)$ for $\forall X, Y$, and satisfies

$$\delta_1(P) \in PUP + PUQ \quad \text{and} \quad \delta_1(Q) \in PUQ + QUQ. \tag{2.6}$$

Thus, by Eq.(2.6) and $\mathbf{P_D}$ for $n = 1$, we have

$$\delta_1(PXQ) = \delta_1(P)PXQ + P\tau_1(PXQ) \in PUQ;$$

$$\delta_1(PXP) = \delta_1(P)PXP + P\tau_1(PXP) \in PUP + PUQ$$

and

$$\delta_1(QXQ) = \delta_1(Q)QXQ + Q\tau_1(QXQ) \in PUP + QUQ.$$

Remark 2.3. By the above argument, for any generalized Jordan higher derivation $D = (\delta_i)_{i \in \mathbb{N}}$ on a triangular ring $\mathcal{U}$, δ_1 is in fact a generalized derivation and also satisfies the following properties:

$\mathbf{P_1}$: (i) $\delta_1(P) \in (PUP + PUQ)$; (ii) $\delta_1(PUQ) \subseteq PUQ$; (iii) $\delta_1(PUP) \subseteq PUP + PUQ$ and $\delta_1(QUQ) \subseteq QUQ + PUQ$; (iv) $\delta_1(XY) = \delta_1(X)Y + X\tau_1(Y)$ for $\forall X, Y$.

3. Characterizations of generalized Jordan Higher Derivations

In this section, we discuss the generalized Jordan higher derivations on triangular rings. The following is our main result.

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings, and $\mathcal{M}$ be a $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let $\mathcal{U} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular ring and P be the standard idempotent of $\mathcal{U}$. Assume that $G = (\delta_i)_{i \in \mathbb{N}}$ an additive generalized Jordan higher derivation of $\mathcal{U}$ and $D = (\tau_i)_{i \in \mathbb{N}}$ the relating Jordan higher derivation. Then for any $X, Y \in \mathcal{U}$ and any $n \in \mathbb{N}$, we have $\delta_n(XY) = \Sigma_{i+j=n}\delta_i(X)\tau_j(Y)$, that is, $G = (\delta_i)_{i \in \mathbb{N}}$ is a generalized higher derivation.*

Proof. We proceed by induction on $n \in \mathbb{N}$. Assume that $G = (\delta_i)_{i \in \mathbb{N}}$ be a generalized Jordan higher derivation of $\mathcal{U}$ and $D = (\tau_i)_{i \in \mathbb{N}}$ the relating Jordan higher derivation.

If $n = 1$, by Remark 2.3, δ_1 is a generalized derivation satisfying $\mathbf{P_1}$. So the theorem is true in this case.

Now suppose that for any $X, Y \in \mathcal{U}$ and any $m < n$, δ_m satisfies the following properties:

$\mathbf{P_m}$: (i) $\delta_m(P) \in (PUP + PUQ)$; (ii) $\delta_m(PUQ) \subseteq PUQ$; (iii) $\delta_m(PUP) \subseteq PUP + PUQ$ and $\delta_m(QUQ) \subseteq QUQ + PUQ$; (iv) $\delta_m(XY) = \sum_{i+j=m} \delta_i(X)\tau_j(Y)$ for $\forall X, Y$.

Our aim is to show that δ_n satisfies the following properties:

$\mathbf{P_n}$: (i)$\delta_n(P) \in (PUP + PUQ)$; (ii) $\delta_n(PUQ) \subseteq PUQ$; (iii) $\delta_n(PUP) \subseteq PUP + PUQ$ and $\delta_n(QUQ) \subseteq QUQ + PUQ$; (iv) $\delta_n(XY) = \sum_{i+j=n} \delta_i(X)\tau_j(Y)$ for $\forall X, Y$.

And therefore, $G = (\delta_i)_{i \in \mathbb{N}}$ be a generalized higher derivation of $\mathcal{U}$. We will prove it by several steps.

Step 1. $\delta_n(P) \in (PUP + PUQ)$.

In fact, since $\delta_i(P) \in (PUP + PUQ)$ for $i = 1, 2, ..., n-1$ and $\tau_i(P) \in PUQ$ for $i = 1, 2, ..., n$, we have $\delta_i(P)\tau_j(P) \in PUQ$ for $i = 1, 2, ..., n-1$ with $i + j = n$ and $P\tau_n(P) \in PUQ$. Hence

$$\delta_n(P) = \sum_{i+j=n} \delta_i(P)\tau_j(P) \in \delta_n(P)P + PUQ,$$

which implies that $\delta_n(P) \in (PUP + PUQ)$.

Step 2. $\delta_n(PUQ) \subseteq PUQ$.

Take any $X \in \mathcal{U}$. By Lemma 2.1(1), we have

$$\begin{aligned}
\delta_n(PXQ) = & \ \delta_n(PPXQ + PXQP) \\
= & \ \textstyle\sum_{i+j=n}(\delta_i(P)\tau_j(PXQ) + \delta_i(PXQ)\tau_j(P)) \\
= & \ \delta_n(P)PXQ + \delta_n(PXQ)P + P\tau_n(PXQ) + PXQ\tau_n(P) \\
& + \textstyle\sum_{i+j=n;i\neq 0,n}(\delta_i(P)\tau_j(PXQ) + \delta_i(PXQ)\tau_j(P)).
\end{aligned}$$

With Step 1 and the properties $\mathbf{P_m}$, $\mathbf{P_D}$, it is clear that $\sum_{i+j=n;i\neq 0,n}(\delta_i(P)\tau_j(PXQ) + \delta_i(PXQ)\tau_j(P)) \in PUQ$, $\delta_n(P)PXQ \in PUQ$, $P\tau_n(PXQ) \in PUQ$ and $PXQ\tau_n(P) = 0$. So we get $\delta_n(PXQ) - \delta_n(PXQ)P \in PUQ$, which implies that $Q\delta_n(PXQ)Q = 0$.

Similarly, using the equation $\delta_n(PXQ) = \delta_n(QPXQ + PXQQ)$, one can get $P\delta_n(PXQ)P = 0$. So $\delta_n(PXQ) = P\delta_n(PXQ)Q \in PUQ$.

Step 3. $\delta_n(PUP) \subseteq PUP + PUQ$.

For any $X \in \mathcal{U}$, by Lemma 2.1(2), we have

$$\delta_n(PXP) = \textstyle\sum_{i+j+k=n} \delta_i(P)\tau_j(PXP)\tau_k(P).$$

By Step 1 and $\mathbf{P_m}$, $\mathbf{P_D}$, for any $i, j, k \in \{0, 1, 2, \cdots, n\}$, we have $\delta_i(P) \in PUP + PUQ$ and $\tau_j(PXP) \in PUP + PUQ$. It follows that

$$\delta_i(P)\tau_j(PXP)\tau_k(P) \in (PUP + PUQ)(PUP + PUQ) = PUP + PUQ,$$

and so $\delta_n(PXP) \in PUP + PUQ$.

By a similar argument to that of Step 3, one can check that

Step 4. $\delta_n(QUQ) \subseteq PUQ + QUQ$.

Step 5. For any $X, Y \in \mathcal{U}$, the following five equations hold:

(1) $\delta_n(PXPYP)Q = \sum_{i+j=n;i\neq n}(\delta_i(PXP)\tau_j(PYP))Q$;

(2) $\delta_n(PXPYQ) = \sum_{i+j=n} \delta_i(PXP)\tau_j(PYQ)$;

(3) $\delta_n(PXQYQ) = \sum_{i+j=n} \delta_i(PXQ)\tau_j(QYQ)$;

(4) $\delta_n(QXQYQ) = \sum_{i+j=n} \delta_i(QXQ)\tau_j(QYQ)$;

(5) $\sum_{i+j=n} \delta_i(PXP)\tau_j(QYQ) = 0$.

In fact, for any $X, Y \in \mathcal{U}$, by Step 2 and $\mathbf{P_D}$, we have

$$
\begin{aligned}
\delta_n(PXPYQ) =\ & \delta_n(PXPYQ + PYQPXP) \\
=\ & \sum_{i+j=n}(\delta_i(PXP)\tau_j(PYQ) + \delta_i(PYQ)\tau_j(PXP)) \\
=\ & \sum_{i+j=n} \delta_i(PXP)\tau_j(PYQ).
\end{aligned}
$$

That is, (2) holds.

Similarly, one can check that (3) and (5) is true.

For (1), we have

$$
\begin{aligned}
\delta_n(PXP) =\ & \sum_{i+j+k=n} \delta_i(P)\tau_j(PXP)\tau_k(P) \\
=\ & \sum_{i+j=n} \delta_i(P)\tau_j(PXP)P + \sum_{i+j=n-1} \delta_i(P)\tau_j(PXP)\tau_1(P) \\
& + \sum_{i+j=n-2} \delta_i(P)\tau_j(PXP)\tau_2(P) + \ldots \\
& + \sum_{i+j=1} \delta_i(P)\tau_j(PXP)\tau_{n-1}(P) + PXP\tau_n(P).
\end{aligned}
$$

By induction, the above equation becomes

$$
\begin{aligned}
\delta_n(PXP) =\ & \sum_{i+j=n} \delta_i(P)\tau_j(PXP)P + \delta_{n-1}(PXP)\tau_1(P) \\
& + \delta_{n-2}(PXP)\tau_2(P) + \ldots + \delta_1(PXP)\tau_{n-1}(P) + PXP\tau_n(P),
\end{aligned}
$$

and so

$$
\begin{aligned}
\delta_n(PXP)Q =\ & \delta_{n-1}(PXP)\tau_1(P)Q + \delta_{n-2}(PXP)\tau_2(P)Q \\
& + \ldots + \delta_1(PXP)\tau_{n-1}(P)Q + PXP\tau_n(P)Q.
\end{aligned}
$$

Thus for any $X, Y \in \mathcal{U}$, we get

$$\delta_n(PXPYP)Q$$
$$= \delta_{n-1}(PXPYP)\tau_1(P)Q + \delta_{n-2}(PXPYP)\tau_2(P)Q$$
$$+ \ldots + \delta_1(PXPYP)\tau_{n-1}(P)Q + PXPYP\tau_n(P)Q$$
$$= \sum_{i+j=n-1} \delta_i(PXP)\tau_j(PYP)\tau_1(P)Q$$
$$+ \sum_{i+j=n-2} \delta_i(PXP)\tau_j(PYP)\tau_2(P)Q$$
$$+ \ldots + \sum_{i+j=1} \delta_i(PXP)\tau_j(PYP)\tau_{n-1}(P)Q + PXPYP\tau_n(P)Q$$
$$= PXP(\tau_{n-1}(PYP)\tau_1(P) + \tau_{n-2}(PYP)\tau_2(P) + \ldots + \tau_1(PYP)\tau_{n-1}(P) + PYP\tau_n(P))Q$$
$$+ \delta_1(PXP)(\tau_{n-2}(PYP)\tau_1(P) + \tau_{n-3}(PYP)\tau_2(P) + \ldots + \tau_1(PYP)\tau_{n-2}(P))Q$$
$$+ \ldots + \delta_{n-2}(PXP)(\tau_1(PYP)\tau_1(P) + PYP\tau_2(P))Q + \delta_{n-1}(PXP)(PYP\tau_1(P))Q$$
$$= PXP\tau_n(PYP)Q + \delta_1(PXP)\tau_{n-1}(PYP)Q$$
$$+ \ldots + \delta_{n-2}(PXP)\tau_2(PYP)Q + \delta_{n-1}(PXP)\tau_1(PYP)Q$$
$$= \sum_{i+j=n; i \neq n}(\delta_i(PXP)\tau_j(PYP))Q.$$

Hence (1) holds.

Finally, we prove (4). For any $X, Y \in \mathcal{U}$, by Lemma 2.1(2), we have

$$\delta_n(QXQ) = \sum_{i+j+k=n} \delta_i(Q)\tau_j(QXQ)\tau_k(Q)$$
$$= \sum_{i+j=n} \delta_i(Q)\tau_j(QXQ)Q + \sum_{i+j+k=n; k\neq 0} \delta_i(Q)\tau_j(QXQ)\tau_k(Q).$$

Note that, by Steps 1, 4 and the properties of τ_i (Remark 2.2),

$$\sum_{i+j+k=n; k\neq 0} \delta_i(Q)\tau_j(QXQ)\tau_k(Q) \in (PUQ + QUQ)(PUQ + QUQ)(PUQ) = \{0\}.$$

Thus we get

$$\delta_n(QXQ) = \sum_{i+j=n} \delta_i(Q)\tau_j(QXQ)Q \quad \text{for all} \ \ X \in \mathcal{U}.$$

Since $\tau_i(QYQ) = \tau_i(QYQ)Q$, the above equation yields

$$\delta_n(QXQYQ) = \sum_{i+j=n} \delta_i(Q)\tau_j(QXQYQ)Q$$
$$= \sum_{i+j=n} \delta_i(Q)(\sum_{p+q=j} \tau_p(QXQ)\tau_q(QYQ))Q$$
$$= \sum_{i+p+q=n} \delta_i(Q)\tau_p(QXQ)\tau_q(QYQ)Q$$
$$= \sum_{s=0}^{n}[\sum_{i+p=s}(\delta_i(Q)\tau_p(QXQ)Q)\tau_{n-s}(QYQ)]Q$$
$$= \sum_{s=0}^{n}[\delta_s(QXQ)\tau_{n-s}(QYQ)]Q$$
$$= \sum_{s=0}^{n} \delta_s(QXQ)\tau_{n-s}(QYQ) = \sum_{i+j=n} \delta_i(QXQ)\tau_j(QYQ).$$

It follows that (4) holds.

Step 6. $\delta_n(XY) = \sum_{i+j=n} \delta_i(X)\tau_j(Y)$ for all $X, Y \in \mathcal{U}$, that is, the theorem is true.

We first prove that $[\delta_n(XY) - \sum_{i+j=n} \delta_i(X)\tau_j(Y)]P = 0$. In fact, for any $X, Y, S \in \mathcal{U}$, by Steps 2-5, on the one hand, we have

$$
\begin{aligned}
\delta_n(XYPSQ) =& \ \delta_n(PXPYPSQ) = \sum_{i+j=n} \delta_i(PXPYP)\tau_j(PSQ) \\
=& \ \delta_n(PXPYP + PXPYQ + PXQYQ + QXQYQ)PSQ \\
& + \sum_{i+j=n; i \neq n} \delta_i(PXPYP)\tau_j(PSQ) \\
=& \ \delta_n(XY)PSQ + \sum_{i+j=n; i \neq n} \delta_i(PXPYP)\tau_j(PSQ).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\delta_n(XYPSQ) =& \ \delta_n(PXPYPSQ) = \sum_{i+j=n} \delta_i(PXP)\tau_j(PYPSQ) \\
=& \ \sum_{i+j=n} \delta_i(PXP) \sum_{p+q=j} \tau_p(PYP)\tau_q(PSQ) \\
=& \ \sum_{i+p+q=n} \delta_i(PXP)\tau_p(PYP)\tau_q(PSQ) \\
=& \ \sum_{i+p=n} \delta_i(PXP)\tau_p(PYP)PSQ + \sum_{i+p+q=n; q \neq 0} \delta_i(PXP)\tau_p(PYP)\tau_q(PSQ) \\
=& \ \sum_{i+p=n} \delta_i(X)\tau_p(PYP)PSQ + \sum_{s+q=n; q \neq 0} \delta_s(PXPYP)\tau_q(PSQ) \\
=& \ \sum_{i+p=n} \delta_i(X)\tau_p(Y)PSQ + \sum_{s+q=n; q \neq 0} \delta_s(PXPYP)\tau_q(PSQ).
\end{aligned}
$$

The last two equations hold since $\delta_i(PXQ+QXQ)\tau_p(PYP) = 0$ and $\tau_p(PYQ+QYQ)PSQ = 0$ for all i, p (by induction on n, Steps 2-3 and the property $\mathbf{P_D}$). Comparing the above two equations, we obtain $[\delta_n(XY) - \sum_{i+j=n} \delta_i(X)\tau_j(Y)]PSQ = 0$ for all $PSQ \in P\mathcal{U}Q$. Since $\mathcal{M}$ is faithful as a left $\mathcal{A}$-module and $Q\mathcal{U}P = \{0\}$, it follows that

$$
[\delta_n(XY) - \sum_{i+j=n} \delta_i(X)\tau_j(Y)]P = 0. \tag{3.1}
$$

We still need to prove that $[\delta_n(XY) - \sum_{i+j=n} \delta_i(X)\tau_j(Y)]Q = 0$. For any $S \in \mathcal{U}$, by Steps 1-5, we have

$$
\begin{aligned}
\delta_n(XYQSQ) =& \ \delta_n(PXPYQSQ) + \delta_n(PXQYQSQ) + \delta_n(QXQYQSQ) \\
=& \ \sum_{i+j=n} \delta_i(PXPYQ)\tau_j(QSQ) + \sum_{i+j=n} \delta_i(PXQYQ)\tau_j(QSQ) \\
& + \sum_{i+j=n} \delta_i(QXQYQ)\tau_j(QSQ) \\
=& \ \delta_n(PXPYP + PXPYQ + PXQYQ + QXQYQ)QSQ \\
& - \delta_n(PXPYP)QSQ + \Delta \\
=& \ \delta_n(XY)QSQ - \delta_n(PXPYP)QSQ + \Delta,
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta \doteq& \ \sum_{i+j=n; i \neq n} \delta_i(PXPYQ)\tau_j(QSQ) + \sum_{i+j=n; i \neq n} \delta_i(PXQYQ)\tau_j(QSQ) \\
& + \sum_{i+j=n; i \neq n} \delta_i(QXQYQ)\tau_j(QSQ).
\end{aligned}
$$

On the other hand,

$$\delta_n(XYQSQ)$$
$$= \delta_n(PXPYQSQ) + \delta_n(PXQYQSQ) + \delta_n(QXQYQSQ)$$
$$= \sum_{i+j=n} \delta_i(PXP)\tau_j(PYQSQ) + \sum_{i+j=n} \delta_i(PXQ)\tau_j(QYQSQ)$$
$$\quad + \sum_{i+j=n} \delta_i(QXQ)\tau_j(QYQSQ)$$
$$= \sum_{i+p+q=n} \delta_i(PXP)\tau_p(PYQ)\tau_q(QSQ) + \sum_{i+p+q=n} \delta_i(PXQ)\tau_p(QYQ)\tau_q(QSQ)$$
$$\quad + \sum_{i+p+q=n} \delta_i(QXQ)\tau_p(QYQ)\tau_q(QSQ)$$
$$= \sum_{i+p=n} \delta_i(PXP)\tau_p(PYQ)QSQ + \sum_{i+p=n} \delta_i(PXQ)\tau_p(QYQ)QSQ$$
$$\quad + \sum_{i+p=n} \delta_i(QXQ)\tau_p(QYQ)QSQ + \sum_{i+p+q=n;q\neq0} \delta_i(PXP)\tau_p(PYQ)\tau_q(QSQ)$$
$$\quad + \sum_{i+p+q=n;q\neq0} \delta_i(PXQ)\tau_p(QYQ)\tau_q(QSQ) + \sum_{i+p+q=n;q\neq0} \delta_i(QXQ)\tau_p(QYQ)\tau_q(QSQ)$$
$$= \sum_{i+p=n} \delta_i(PXP + PXQ + QXQ)\tau_p(PYP + QYQ + PYQ)QSQ$$
$$\quad - \sum_{i+p=n} \delta_i(PXP)\tau_p(PYP)QSQ - \sum_{i+p=n} \delta_i(PXP)\tau_p(QYQ)QSQ$$
$$\quad - \sum_{i+p=n} \delta_i(PXQ)\tau_p(PYP)QSQ - \sum_{i+p=n} \delta_i(PXQ)\tau_p(PYQ)QSQ$$
$$\quad - \sum_{i+p=n} \delta_i(QXQ)\tau_p(PYP)QSQ - \sum_{i+p=n} \delta_i(QXQ)\tau_p(PYQ)QSQ + \Delta$$
$$= \sum_{i+p=n} \delta_i(X)\tau_p(Y)QSQ - \sum_{i+p=n} \delta_i(PXP)\tau_p(PYP)QSQ + \Delta$$
$$= \sum_{i+p=n} \delta_i(X)\tau_p(Y)QSQ - \delta_n(PXPYP)QSQ + \Delta.$$

Comparing the above two equations, we get $[\delta_n(XY) - \sum_{i+j=n} \delta_i(X)\tau_j(Y)]QSQ = 0$ for all $S \in \mathcal{U}$. It follows that

$$[\delta_n(XY) - \sum_{i+j=n} \delta_i(X)\tau_j(Y)]Q = 0. \tag{3.2}$$

Combining Eq.(3.1) and (3.2), we get $\delta_n(XY) = \sum_{i+j=n} \delta_i(X)\tau_j(Y)$ for all $X, Y \in \mathcal{U}$. The proof is complete. $\qquad\square$

Let $F = (\delta_i)_{i\in\mathbb{N}}$ be any generalized Jordan triple higher derivation of $\mathcal{U}$ and $D = (\tau_i)_{i\in\mathbb{N}}$ the relating Jordan triple higher derivation. It is easy to check that $\tau_i(I) = 0 (i = 1, 2, ..., n)$. So it is obvious from Lemma 2.1(2) that F is also a generalized Jordan higher derivation of $\mathcal{U}$. Hence the following theorem is immediate.

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings, and $\mathcal{M}$ be a $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular ring. Then every generalized Jordan triple higher derivation of $\mathcal{U}$ is a generalized higher derivation.*

Recall that a nest $\mathcal{N}$ on a Banach space X is a chain of closed subspaces of X which is closed under the formation of arbitrary closed linear span and intersection, and which includes $\{0\}$ and X. The nest algebra associated to the nest $\mathcal{N}$, denoted by $\mathrm{Alg}\mathcal{N}$, is the weakly closed operator algebra consisting of all operators that leave $\mathcal{N}$ invariant, i.e.,

$$\mathrm{Alg}\mathcal{N} = \{T \in \mathcal{B}(X) : TN \subseteq N \text{ for all } N \in \mathcal{N}\}.$$

If X is a Hilbert space, then every $N \in \mathcal{N}$ corresponds to a projection P_N satisfying $P_N = P_N^* = P_N^2$ and $N = P_N(X)$. However, it is not always the case for general nests on Banach spaces as $N \in \mathcal{N}$ may be not complemented. We refer the reader to [4] for the theory of nest algebras.

As an application of Theorem 3.1 and 3.2 to the nest algebra case, we have

Theorem 3.3. *Let $\mathcal{N}$ be a nest on a Banach space X and there exists a non-trivial element in $\mathcal{N}$ which is complemented in X (in particular, $\mathcal{N}$ be any nest on a Hilbert space H). Then every generalized Jordan (triple) higher derivation on $\mathrm{Alg}\mathcal{N}$ is a generalized higher derivation.*

Remark 3.4. By Theorem 3.1 and 3.2, we show that the concepts of generalized Jordan triple higher derivation and generalized Jordan higher derivation on triangular algebras are equivalent, and so the concepts of generalized Jordan triple higher derivation, generalized Jordan higher derivation and generalized higher derivation on triangular rings are equivalent to each other.

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